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THE DYNAMIC MODEL OF ADVERTISING COSTS

Abstract. *One of the main problems that any company should solve for implementation of its advertising strategy is to find an optimal distribution of advertising costs over the planning period under a limited advertising budget. In order to find the solution of this problem, it is necessary to analyze statistical data, identify character of influence of advertising costs and non-advertising factors on the company's financial result (profit or loss).*

In this paper the dynamic model of optimal control of advertising expenses is constructed. In contrast to classical dynamic advertising models (Nerlove-Arrow, Vidale-Wolfe etc.), the model describes accumulated advertising effect and accumulated effect of previous sales taking into account delayed consumers' reaction on advertisements. The optimal control problem is formulated as a system of nonlinear integral equations of Volterra type and an integral criterion functional.

The special case of the nonlinear integral equation of Volterra type, which is produced by typically boundaries, is formulated. The theorem of existence of solution of this equation is proved. Also the theorem of solution existence of the total profit maximization problem for the planning period under restrictions is examined. These restrictions involve the limited advertising budget and the functional relation between advertising and consumers' reaction.

Keywords: *advertising; mathematical model of advertising; optimal control; existence of solution; necessary conditions.*

JEL Classification: C41, C53, C61, C62, M37

1. Introduction

Generally, the advertising costs should be included in the total costs. However, the main purpose of advertising is to stimulate consumer demand without changing of production technologies or quality of goods, so it makes sense to consider advertising separately from total costs.

The advertising effect is not instantaneous. A time interval usually appears between the moment of posting of advertising and consumers' reaction.

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Also there are non-advertising factors that influence on sales, i.e. quality of goods, business reputation etc. These factors make the effect of previous sales and encourage consumers to repeat their previous purchases.

Therefore, consumer demand is changing under influence of the accumulated effect of previous advertising costs and the accumulated effect of previous sales.

It is logical to assume that after some time the effect of the first advertisement becomes weaker than next one. Similarly, the experience of the first purchase is slowly forgotten and it has little influence on next purchases. So it is possible to consider fixed intervals, on which the influence of the accumulated advertising effect and the accumulated effect of previous sales are the most significant.

The advertising budget of the company is usually limited to the amount of budget that can be spent on implementation of the advertising strategy.

Thus, the problem of the best distribution of advertising costs for the planning period can be considered as the optimal control problem under some restrictions: specificity of the impact of advertising and non-advertising factors on current sales and the limiting value of the advertising budget.

The dynamic optimization models were considered by different authors [12], [3], [10], [2], [7], [1]. These models are widely used and are the basis of many modern advertising models. Vidal and Wolf developed the model based on the differential equation, which describes change dependence in sales according to four factors: advertising costs, market saturation level, an exponential sales decay constant, a response constant (1957). Nerlove and Arrow constructed the classical N-A model to describe the time-dependent demand in a general function form. Specifically, the consumer demand depends on the goodwill stock that involves the impacts of current and past advertising costs (1962). Kimball applied the dynamic Lanchester model of advertising, which is similar to Lanchester’s models of warfare. He analyzed advertising strategies of two firms for a duopoly setting (1957). Also, most of modern advertising models are discussed by Jian Huang, Mingming Leng, Liping Liang (2011).

In our paper, in addition to advertising costs, several factors of influence on consumer demand are taken into account. The delayed consumer reaction is considered here too. The paper is expansion of the research [9], and it includes the practical application. It describes implementation of the dynamic model of advertising costs in detail.

2. State of the Problem

Denote $x(t)$ as the company's revenue which is the monetary expression of the consumer demand at the moment t, $u(t)$ as the advertising costs at the moment t, $v(t)$ and $y(t)$ as the functions which determine respectively accumulated advertising impact and accumulated influence of previous sales on current sales at the moment t :

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$$v(t) = \int_{\tau_{1u}}^{\tau_{2u}} G_u(\tau)u(t-\tau)d\tau \; ; \qquad (1)$$

$$y(t) = \int_{\tau_{1x}}^{\tau_{2x}} G_x(\tau)x(t-\tau)d\tau \qquad (2)$$

The revenue at the moment *t* depends on *v*(*t*) and *y*(*t*):

$$x(t) = f(v(t), y(t)). \qquad (3)$$

In (2) τ_{1u} and τ_{2u} are boundaries of the time interval on which the advertising impact is accumulated, τ_{1x} and τ_{2x} are boundaries of the time interval on which the impact of previous sales is accumulated, $G_u(\tau)$ and $G_x(\tau)$ are the functions which determine the effect of previous advertising costs and previous sales, respectively.

The identification of the functions $f(v, y)$, $G_u(\tau)$, $G_x(\tau)$ is the problem of econometric analysis. Let's make some assumptions about these functions:

1) If the variables *v* and *y* take a small values (a market isn't saturated by company's products and advertising is viewed by consumers positively), the function $f(v, y)$ increases. But consumers' reaction may change over time [1] and the function $f(v, y)$ may become non-increasing with respect to *v*. Taking into account the dependence of $f(v, y)$ on the variable *y*, it is possible to make an assumption that $f(v, y)$ is decreasing because of market saturation and supply constraints of the firm. We can consider the function $f(v, y)$ as concave with respect to *y*.

2) Let us suppose the following properties of the function $G_u(\tau)$: up to the certain moment τ^* the advertising influence on the demand increases. After that the advertising effect begins to decrease until it disappears; the possibility of negative advertising effect is excluded, i.e. advertising costs of the company don't reduce product demand. Thus, the function $G_u(\tau)$ is non-negative and it has only one local (also global) maximum τ^*, which is the moment of the highest advertising effect. If this function is differentiable, then the assumption is equivalent to the following conditions:

$$G_u(\tau) \geq 0, \forall \tau \in [0;+\infty); \quad \lim_{\tau \to +\infty} G_u(\tau) = 0,$$

$$G'_u(\tau) \geq 0, \tau \in [0;\tau_u^*); \quad G'_u(\tau) \leq 0, \quad \tau \in (\tau_u^*;+\infty).$$

3) Regarding the function $G_x(\tau)$ we can note that consumers repeat purchases taking into account their previous experience. In this case consumer

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demand is getting higher and the function $G_X(\tau)$ is increasing. However, the experience of the first purchases is usually forgotten. It influences on the current purchases weakly, giving place to the recent experience. Therefore, the function $G_X(\tau)$ has the properties:

$$G_x(\tau) \geq 0, \forall \tau \in [0;+\infty); \quad \lim_{\tau \to +\infty} G_x(\tau) = 0,$$

$$G'x(\tau) \geq 0, \tau \in [0;\tau_x^*); \quad G'x(\tau) \leq 0, \quad \tau \in (\tau_x^*;+\infty).$$

The financial result of the company is the profit or loss. Obviously, at each point t the profit determined by the condition $\pi(x(t),(u(t)) = x(t) - c(x(t),t)$ where $c(x(t),t)$ is the total costs which are associated with revenue $x(t)$ at the moment t. The function of the total costs includes fixed and variable costs associated with the process of production and selling. Having designated the total profit on the planning interval $[0;T]$ as $\Pi(x,u)$ we can write the objective functional:

$$\Pi(T) = \int_0^T \pi(x(t),u(t))dt = \int_0^T (f(v(t), y(t)) - u(t) - c(x(t),t))dt$$

where $c(x(t),t) = c_1(x(t),t) + c_2$, c - total costs of the firm, associated with the production of goods excepting the advertising costs, c_1 - variable costs, c_2 - fixed costs.

It is reasonable to assume that variable costs are in direct ratio to the volume of output with the depreciation rate μ. In this case we have following:

$$\Pi(T) = \int_0^T ((1-\mu)(f(v(t), y(t)) - u(t))dt. \tag{4}$$

Companies have different approaches to determining the advertising budget, e.g. fixed amount per unit time

$$0 \leq u(t) \leq b, \quad t \in [0;T]. \tag{5}$$

Let us suppose that the functions of revenue $\hat{x}(t)$ and advertising expenses $\hat{u}(t)$ are known before the beginning of the planning period. So, we can formulate the following optimization problem: to maximize the functional (4) under the conditions (1), (2), (3), (5).

Transform the problem (1), (2), (3), (4), (5) to the optimal control problem with the integral equations of Volterra type. Define the functions $\phi_u(t)$, $\phi_x(t)$, $\overline{G_u}(t,s)$, $\overline{G_x}(t,s)$:

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$$\phi_u(t) = \begin{cases} \int_{t-\tau_{2u}}^{t-\tau_{1u}} G_u(t-s)\tilde{u}(s)ds, & 0 \le t \le \tau_{1u}, \\ \int_{t-\tau_{2u}}^{0} G_u(t-s)\tilde{u}(s)ds, & \tau_{1u} \le t \le \tau_{2u}, \\ 0, \quad \tau_{2u} \le t; \end{cases}$$

$$\phi_x(t) = \begin{cases} \int_{t-\tau_{2x}}^{t-\tau_{1x}} G_x(t-s)\tilde{x}(s)ds, & 0 \le t \le \tau_{1x}, \\ \int_{t-\tau_{2x}}^{0} G_u(t-s)\tilde{x}(s)ds, & \tau_{1x} \le t \le \tau_{2x}, \\ 0, \quad \tau_{2x} \le t; \end{cases}$$

$$\overline{G_u}(t,s) = \begin{cases} Gu(t,s), & \tau_{1u} \le t-s \le \tau_{2u}, \\ 0, & else; \end{cases}$$

$$\overline{G_x}(t,s) = \begin{cases} Gx(t,s), & \tau_{1x} \le t-s \le \tau_{2x}, \\ 0, & else. \end{cases}$$

Here $\tilde{x}(t)$, $\tilde{u}(t)$ are the functions that respectively describe the company's revenue variation and the advertising costs variation before the beginning of the planning period.

Using the introduced functions, the total profit and the accumulated effects of previous sales and advertising costs can be represented as:

$$v(t) = \phi_u(t) + \int_0^t \overline{G_u}(t-s)u(s)ds, \tag{6}$$

$$y(t) = \phi_x(t) + \int_0^t \overline{G_x}(t-s) f(v(s), y(s))ds. \tag{7}$$

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$$\Pi(t) = \int_0^t ((1-\mu)(f(v(s), y(s)) - u(s))ds\,. \qquad (8)$$

The maximizing $\Pi(T)$ under the conditions (5), (6), (7), (8) presents the optimal control problem with the integral equations of Volterra type.

3. Existence of the Solution

Let us suppose that advertising costs function *u*(*t*) is piecewise-continuous on the right in [t_0; T]. Continuity of the functions $G_u(\tau)$, $G_x(\tau)$ in $[\tau_{1u};\tau_{2u}]$, $[\tau_{1x};\tau_{2x}]$ follows from researchers' requirements to the function properties.

Consider the problem of the solution existence of equations (6), (7), (8).

If $G_u(\tau)$ is continuous, then $\overline{G_u}(\tau)$ is piecewise-continuous, *v*(*t*) is continuous in [t_0; T]. Considering the equations (5), (6) we can state existence of the value b_1>0:

$$0 \le \phi_u(t) + \int_0^t G_u(t-s)u(s)ds \le \max_{0\le t\le\tau_{1u}}\left(\phi_u(t) + \int_0^t G_u(t-s)bds\right) \le b_1.$$

Thus, the function of the accumulated advertising effect satisfies the condition $0 \le v(t) \le b_1$ for any advertising strategy *u*(*t*) that satisfies the condition (5). The problem of the solution existence of the equation (7) is solved by the theorem 1.

Theorem 1. *Let the functions* $G_x(\tau) \in C([\tau_{1u};\tau_{2u}])$, $G_x(\tau) \in C([\tau_{1u};\tau_{2u}])$, $\hat{u}(t)$, $\hat{x}(t)$ *are piecewise-continuous under* $t \le 0$, *the function* $f(v,y)$ *is continuous and it satisfies the Lipschitz condition with respect to y for all* y. *Then, for any piecewise-continuous function u*(*t*), $t \in [0;T]$ *that satisfies the restriction (5), there exists continuous and unique function y*(*t*), $t \in [0;T]$ *that satisfies the condition (7).*

Proof. The proof of the solution existence of the linear equation of Volterra type is described in [6]. Based on this statement we prove a solution existence of the integral equation (7).

Initially, the function $f(v,y)$ satisfies the Lipschitz condition with respect to *y*. So, there exists the constant L: $|f(v,y_1) - f(v,y_1)| \le L|y_1 - y_2|$, $\forall y_1, y_2$. Define the operator A in the following form:

$$Ay(t) \equiv \phi_X(t) + \int_0^t \overline{G_X}(t-s) f(y(s), v(s))ds\,.$$

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There is the finite number $M = \max_{\tau_{1x} \le \tau \le \tau_{2x}} G_x(\tau)$. Due to properties of the function $\overline{G_x}(\tau)$, there exists $\max_{\tau_{1x} \le \tau \le \tau_{2x}} \overline{G_x}(\tau)$ which is equal to *M*. So, we can write the following inequality:

$$\left|A_{y_1}(t) - A_{y_2}(t)\right| = \left|\int_0^t \overline{G_X}(t-s)(f(v(s), y_1(s)) - f(v(s), y_2(s)))ds\right| \le$$
$$\le ML \max_{0 \le s \le T} |y_1(s) - y_2(s)|.$$

Introduce A^k , *k* is the multiple successive applying of the operator *A*. In this case $A^2 y \equiv A(Ay)$, $A^k y \equiv A(A^{k-1} y)$. So we have the inequality:

$$\left|A^2{}_{y_1}(t) - A^2{}_{y_2}(t)\right| = \left|\int_0^t \overline{G_X}(t-s)(f(v(s), Ay_1(s)) - f(v(s), Ay_2(s)))ds\right| \le$$
$$\le ML \int_0^t \left|A_{y_1}(t) - A_{y_2}(t)\right| ds \le \frac{(MLt)^2}{2} \max_{0 \le s \le T} |y_1(s) - y_2(s)|.$$

Similarly:

$$\left|A^k{}_{y_1}(t) - A^k{}_{y_2}(t)\right| \le \frac{(MLt)^k}{k!} \max_{0 \le s \le T} |y_1(s) - y_2(s)|.$$

Using the metric in the space of continuous functions $\rho(y_1, y_2) = \max_{0 \le s \le T} |y_1(s) - y_2(s)|$ it is possible to write:

$$\rho(A^k y_1, A^k y_2,) \le \frac{(MLt)^k}{k!} \rho(y_1, y_2).$$

Evidently, there exists k: $\frac{(MLt)^k}{k!} < 1$. It means that operator A^k *is* contracting. Thus, the solution *y*(*t*) of the equation (7) exists, it is unique and continuous in [0; *T*].

Theorem 1 gives the conditions of the existence of the global solution of the equation (7).

Remark 1. Let us suppose that the $f(v, y)$ *is non-negative, concave and non-decreasing monotonously with respect to y. If the finite partial derivative* f'_y *exists with respect to y = 0 and the function f(y; v) satisfies the Lipschitz condition*

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with respect to y for any y. Then the solution of the equation (7) exists and it is non-negative in [0;T].

Consider the optimization problem: to maximize $\Pi(T)$ under the conditions (5), (6), (7), (8).

A finite value of the accumulated profit $\Pi(T)$ is a functional with the control $u(\cdot)$. Introduce $J(u(\cdot)) \equiv \Pi(T)$. Formulate the theorem of the solution existence of the optimization dynamic problem.

Theorem 2. *Assume that the conditions of the theorem 1 are fulfilled, $f(v,y)$ doesn't decrease monotonously with respect to v; then there are two alternatives:*

1. *There exists $\{u^*(t), 0 \le t \le T\}$: (5), a solution of the equations (6), (7), (8) and a value $\overline{J}: J(u^s(\cdot)) \to \overline{J}$, $s \to \infty$, $J(u(\cdot)) \le \overline{J}$ for any $u(\cdot)$: (5).*
2. *There exists a sequence of control functions $\{u^s(t), 0 \le t \le T\}$: (5) and a value $\overline{J}$: $J(u^s(\cdot)) \to \overline{J}$, $s \to \infty$, $J(u(\cdot)) \le \overline{J}$ for any $u(\cdot)$: (5).*

Proof. Let us estimate a solution of the equation (7):

$$y(t) = \phi_x(t) + \int_0^t \overline{G_x}(\tau) f(v(s), y(s)))ds \le$$

$$\le \phi_x(t) + \int_0^t \overline{G_x}(\tau) f(b_1, y(s)))ds = y_{b_1}(t).$$

Here $y_{b_1}(t)$ is a solution of the equation (7): $v(s) \equiv b_1$.

Thus, for the advertising strategy (5) the accumulated influence of previous sales $y(s)$ is limited by a constant K: $y(t) \le K = \max_{0 \le t \le T} yb_1(t)$.

It is possible to demonstrate that the total profit is limited:

$$\Pi(T) = \int_0^T (1-\mu)(f(v(s), y(s)) - u(s))ds \le$$

$$\le \int_0^T (f(v(s), y(s)) - u(s))ds \le T \max_{(v,y)\in D} f(v,y),$$

where $D = \{(v,y): 0 \le y \le K, 0 \le v \le b_1\}$. So, the range of functional $J(u(\cdot))$ of the optimal control problem *(5), (6), (7), (8)* is limited. Denote this range as *L*.

Let us suppose that $\overline{J} = \sup L$. Evidently, $\overline{J}$ exist and is limited. If $\overline{J} \in L$, then first alternative is realized else the second alternative is realized [7].

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Remark 2. If the second alternative of the Theorem 2 is realized, then there exists an approximated solution of the optimal control problem (5), (6), (7), (8) (regarding the value of criterion function). Actually, for any ε>0 there exists such a control function $u_{\varepsilon}(\cdot)$*: (5) and appropriate solutions of (6), (7), (8) that* $\bar{J} - J(u_{\varepsilon}(\cdot)) < \varepsilon$.

4. Practical application

In order to test the dynamic model of advertising, a firm's monthly data set of advertising costs and revenue (in Russian rubles) are analyzed from the period January 2009 to July 2014. The firm is producer of ready-made cloths and located in Ulyanovsk, Russia.

Correlation analysis is made in order to identify the relation between factors $x(t)$ and $u(t-\tau)$, $x(t)$ and $x(t-\tau)$. As a result, the lower and upper limits of the integrations (1), (2) are found in accordance with the highest values of the correlation coefficients. For $\tau_{1u} = 0$ $corr(x(t),u(t)) = 0.74$, for $\tau_{1x} = 0.98$ $corr(x(t),x(t\text{-}1)) = 1$, for $\tau_{2u} = 2$ $corr(x(t),u(t\text{-}2)) = 0.7$, for $\tau_{2x} = 3$ $corr(x(t),x(t\text{-}3)) = 0.93$.

Let us assume that the function $f(v,y)$ is linear with v and y: $f(v,y) = y + v$. Then the equation (3) can be represented:

$$x(t) = \int_1^3 G_x(\tau)x(t-\tau)d\tau + \int_0^2 G_u(\tau)u(t-\tau)d\tau .$$

Basing on the properties of the functions $G_u(\tau)$, $G_x(\tau)$, it is possible to represent them as the following:

$$G_u(\tau) = \exp(a_u\tau^2 + b_u\tau) \ ,$$
$$G_x(\tau) = \exp(a_x\tau^2 + b_x\tau) .$$

The parameters g_{1u}, g_{2u}, g_{1x}, g_{2x} are assessed by applying the lest square method and their assessments were obtained: $\hat{a}_u = \text{-}0.392689003$, $\hat{b}_u = 1.913880196$, $\hat{a}_x = \text{-}0.100000888$, $\hat{b}_x = 1.001288713$.

The optimal control problem is solved by using modified method of local variation.

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4.1 Modified method of local variations

Let us consider the following algorithm. Divide the planning period $[t_0;T]$ on N segments. Define as $\Delta t = \frac{T - t_0}{N}$ ($t_i = t_0 + i \cdot \Delta t$, $i = 0,1,...,N$) the length of each segment, as h the value which is small in comparison to b. Define arrays of advertising costs, values of revenue and functional (4) as $U = [u_0\ u_1\ ...\ u_N]$, $X = [x_0\ x_1\ ...\ x_N]$, $\overline{\Pi} = [\overline{\pi}_0\ \overline{\pi}_1\ ...\ \overline{\pi}_N]$.

Take the first approximation of advertising costs $U_0^0 = [u_0^0\ u_1^0\ ...\ u_N^0]$ where the upper index denotes the iteration number and the lower – the number of elements that will be changed.

Beginning with the first approximation, vary each element of U_i^j successively: $u_{i+}^j = u_i^j + h$, $u_{i-}^j = u_i^j - h$ where h – the value which is small in comparison to *b*. Let us suppose that U_{i+}^j and U_{i-}^j are sets of elements which are equal to the appropriate elements of U_i^j, except u_{i+}^j and u_{i-}^j. The arrays of the functional are defined as $\overline{\Pi}_i^j, \overline{\Pi}_{i+}^j, \overline{\Pi}_{i-}^j$. It is possible to find approximate values of (8) and (4) for each of arrays U_i^j, U_{i+}^j, U_{i-}^j by using one of numerical methods, e.g. the trapezoidal rule:

$$x[i] \approx \Delta t\left(\frac{1}{2}\left(G_X(1)x[i-\frac{1}{\Delta t}] + G_X(3)x[i-\frac{3}{\Delta t}]\right) + \sum_{k=1}^{2/\Delta t-1}\left(G_X(1+k\Delta t)x[i-\frac{1}{\Delta t}-k]\right)\right) + \\ + \Delta t\left(\frac{1}{2}\left(G_u(0)u[i] + G_u(2)u[i-\frac{2}{\Delta t}]\right) + \sum_{k=1}^{2/\Delta t-1}\left(G_u(k\Delta t)u[i-k]\right)\right), \qquad (9)$$

$$\overline{\Pi}(T) \approx \Delta t\left(\frac{1}{2}(\overline{\pi}[0] + \overline{\pi}[N]) + \sum_{i=1}^{N-1}\overline{\pi}[i]\right), \\ \overline{\pi}[i] = x[i] - u[i],\ i = 0,1,...,N. \qquad (10)$$

Choose from the sets of U_i^j, U_{i+}^j, U_{i-}^j the one that gives the maximum value of (10) and denote it as $U_{i\max}^j$. Fix the element with number *i*. The next element with number *i*+1 is varied analogously.

After all of N elements of a particular iteration are checked, the next approximation U_0^{j+1} is taken to be equal to $U_{N\max}^j$.

The search for a solution will be stopped if the required accuracy ε is attained:

$$|\max\{\overline{\Pi}_N^j; \overline{\Pi}_{N+}^j; \overline{\Pi}_{N-}^j\} - \max\{\overline{\Pi}_N^{j+1}; \overline{\Pi}_{N+}^{j+1}; \overline{\Pi}_{N-}^{j+1}\}| \le \varepsilon.$$

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4.2 Results of the experiment

Let the experimental period is one year (or 12 months). The first approximation U^0 includes the actual data of the last year. In this case we can observe how the actual data will change if the model is applied. The limited number of advertising budget b=54335 Russian rubles (maximal advertising costs in according to the data of the last year), the value h=100, the accuracy ε=0.0001. Integral parts of (6), (7), (8) are calculated by the trapezoidal rule. The integrals are divided into N segments equaled to Δt.

The solution is found at the different Δt. In each case the solution has the same structure:

$$u(t) = \begin{cases} b, & 0 \le t \le t^*, \\ 0, & t^* \le t \le T, \end{cases}$$

where t^* is the control switching point.

The table 1 demonstrates the obtained total profit and the control switching point at different Δt.

Table 1. The results of the experiment at the different Δt

Δt	$\Pi(T)$ (Russian rubles)	t* (month)
0.1	$2.88826 \cdot 10^8$	11.92
0.05	$2.11906 \cdot 10^8$	11.96
0.025	$1.96675 \cdot 10^8$	11.96
0.0125	$1.92915 \cdot 10^8$	11.96

It is logical that the value of the total profit changes and becomes more accurate while Δt is decreasing. Eventually, referring to the results, it is fair to say that iteration converges to a solution. The control switching point t^* varies slightly and has a finite value.

In order to check stability of the solution, each of the parameter estimators varies by 5% while the others are unchanged. The results are available in the table 2.

Table 2. Variation of the parameters and the results

parameters				Δt=0.1		Δt=0.05		Δt=0.025		Δt=0.0125	
a_u	a_y	b_u	b_y	t*	Π(T)	t*	Π(T)	t*	Π(T)	t*	Π(T)
$1.05\hat{a}_u$	$\hat{a}_y$	$\hat{b}_u$	$\hat{b}_y$	11.92	$2.7575 \cdot 10^8$	11.96	$2.01311 \cdot 10^8$	11.96	$1.8661 \cdot 10^8$	11.96	$1.82984 \cdot 10^8$
$0.95\hat{a}_u$	$\hat{a}_y$	$\hat{b}_u$	$\hat{b}_y$	11.92	3.05336	11.96	2.25281	11.96	2.09382	11.96	2.05453

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parameters				Δt=0.1		Δt=0.05		Δt=0.025		Δt=0.0125	
a_u	a_y	b_u	b_y	t*	П(T)	t*	П(T)	t*	П(T)	t*	П(T)
					$\cdot10^8$		$\cdot10^8$		$\cdot10^8$		$\cdot10^8$
$\hat{a}_u$	$1{,}05\hat{a}_y$	$\hat{b}_u$	$\hat{b}_y$	11.92	6.12759 $\cdot10^7$	11.96	5.50048 $\cdot10^7$	11.96	5.35033 $\cdot10^7$	11.96	5.31006 $\cdot10^7$
$\hat{a}_u$	$0{,}95a_y$	$\hat{b}_u$	$\hat{b}_y$	11.92	2.42679 $\cdot10^{10}$	11.96	1.30418 $\cdot10^{10}$	11.96	1.10742 $\cdot10^{10}$	11.96	1.05936 $\cdot10^{10}$
$\hat{a}_u$	$\hat{a}_y$	$1{,}05\hat{b}_u$	$\hat{b}_y$	11.92	3.16796 $\cdot10^8$	11.96	2.34557 $\cdot10^8$	11.96	2.18199 $\cdot10^8$	11.96	2.14154 $\cdot10^8$
$\hat{a}_u$	$\hat{a}_y$	$0{,}95b_u$	$\hat{b}_y$	11.92	2.67673 ·108	11.96	1.94827 $\cdot10^8$	11.96	1.80445 $\cdot10^8$	11.96	1.769 $\cdot10^8$
$\hat{a}_u$	$\hat{a}_y$	$\hat{b}_u$	$1{,}05\hat{b}_y$	11.92	1.69079 $\cdot10^{10}$	11.96	8.73804 $\cdot10^{10}$	11.96	7.35149 $\cdot10^{10}$	11.96	7.01813 $\cdot10^{10}$
$\hat{a}_u$	$\hat{a}_y$	$\hat{b}_u$	$0{,}95b_y$	11.92	6.44588 $\cdot10^7$	11.96	5.79974 $\cdot10^7$	11.96	5.64479 $\cdot10^7$	11.96	5.6031 $\cdot10^7$

As we can see, variations of the parameters a_u and b_u do not cause considerable change in the result and we can say that they are stable. But the parameter estimators $\hat{a}_y$ and $\hat{b}_y$ have a noticeable effect on the result, thus estimation of these parameters should be the most accurate to avoid mistakes.

5. Conclusion

In the practical application of the model (1), (2), (3), (4), (5), some difficulties can appear. They are related to properties of the function $f(v,y)$ which ensure existence of a solution of the optimal control problem. Particularly, the multiplicative function $f(v,y)=\beta_0 v^{\beta_1} y^{\beta_2}$ is continuous for any $\beta_1>0$, $\beta_2>0$ in $\{(v,y): y\geq 0, v\geq 0\}$, but in the case when $0<\beta_1<1$, the Lipschitz condition is not fulfilled for $y=0$.

There is a special problem for a researcher: to identify the function type $f(v,y)$ which accords to empirical assumptions and requirements of Theorems 1, 2. Specifically, in the practical application the function $f(v,y)=\alpha_1 v+\alpha_2 y$ is continuous and linear in v and satisfies Lipschitz condition in y. Thus, the optimal control problem has a solution in case when the function $f(v,y)$ is linear.

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REFERENCES

[1] **Berndt, E. R (1996),** ***The Practice of Econometrics: Classic and Contemporary*****;** *Addison-Wesley*, p. 702;
[2] **Dyhta, V. A., Samsonjuk, O. N. (2000).** ***Optimal'noe impul'snoe upravlenie s prilozhenijami. [The Optimal Impulse Control with Applications*****];** Moscow, Fizmatlit, p.256;
[3] **Erickson, G. M. (2009),** ***Advertising Competition in a Dynamic Oligopoly with Multiple Brands***. *Operations Research*, 57, 1106-1113;
[4] **Jian Huang, Mingming Ltng, Liping Liang (2011),** ***Recent Developments in Dynamic Advertising Research***. *European Journal of Operational Research*, 220, 591–609;
[5] **Kimball, G. E. (1957),** ***Some Industrial Applications of Military Operations Research Methods***. *Operations Research*, 5, 201-204;
[6] **Krasnov, M. L. (1975),** ***Integral'nye uravnenija [The Integral Equations***], Moscow, *Nauka*, p.302;
[7] **Krotov, V. F., Lagosha, B. A., Lobanov, S. M., Danilina N. I., Sergeev, S. I. (1990),** ***Osnovy teorii optimal'nogo upravlenija: Ucheb. Posobie dlja jekon. Vuzov. [The Fundamentals of the Control Problem Theory***], Moscow, Vysh.shk., p. 431;
[8] **Lutoshkin, I. V. (2010),** ***Modeling a Response Function to Frequency of Advertising***. *Applied econometric*, 19, 101-111;
[9] **Lutoshkin, I. V., Yamaltdinova, N. R. (2016),** ***The Existence of the Solution to the Control Problem of Advertising Expenses with Distributed Lag***. *The Bulletin of Irkutsk State University*,18, 48-59;
[10] **Mosca, S., Viscolani, B. (2004),** ***Optimal Goodwill Path to Introduce a New Product***. *Journal of Optimization Theory and Applications*, 123, 149-162;
[11] **Nerlove, M. and Arrow, K. J. (1962),** ***Optimal Advertising Policy under Dynamic Conditions***. *Economica*, 29, 129-142;
[12] **Sethi, S. P., Prasad, A., He, X. (2008),** ***Optimal Advertising and Pricing in a New-Product Adoption Model***. *Journal of Optimization Theory and Applications*, 139, 351-360;
[13] **Vidale, M. L., Wolfe, H. B. (1957),** ***An Operations-research Study of Sales Response to Advertising***. *Operations Research*, 5, 370-381.

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